4 Complex Numbers and Quadratic Equations

The voltage produced by a battery is characterized by one real number (called potential), such as +12 volts or –12 volts. But the "AC" voltage in a home requires two parameters. One is a potential, such as 120 volts, and the other is an angle (called phase). The voltage is said to have two dimensions. A 2-dimensional quantity can be represented mathematically as either a vector or as a complex number (known in the engineering context as phasor).

Topic Notes

- *Complex Numbers and its Planar Representation*

COMPLEX NUMBERS AND ITS PLANAR REPRESENTATION 1

TOPIC 1

COMPLEX NUMBERS

"A complex number is a combination of real and imaginary numbers which can be represented as $a + ib$ where a and b are real numbers". Such expressions are known as a cartesian form of the complex number.

Here, a is the real part and b is the imaginary part.

So, $z = a + ib$ is a complex number.

Purely Real and Purely Imaginary Complex Numbers

A complex number $z = x + iy$ is a purely real if its imaginary part is 0, *i.e.* Im(z) = 0 and purely imaginary if its real part is 0 *i.e.* Re(z) = 0.

Equality of Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal, if $x_1 = x_2$ and $y_1 = y_2$.

i.e. $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Thus, two complex numbers are equal if and only if, their real parts are equal and their imaginary parts are also equal.

⚠ Caution

Other relation 'greater than' and 'less than' are not defined for complex number.

Imaginary Numbers

"An imaginary number is a number whose square is less than or equal to zero".

e.g. $\sqrt{-1}, \sqrt{-2}$ .... etc.

Here, $\sqrt{-1}$ can be defined as i (*iota*).

Integral power *i*(*iota*)

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i^4 \times i = i$
$i^6 = i^4 i^2 = -1$
$i^7 = i^4 i^3 = -i$
$i^8 = i^4 . i^4 = 1$

and so on.

Power of i: $i^1\ i^2\ i^3\ i^4\ i^5\ i^6\ i^7$ ....

Simplified form: $i, -1, -i, 1, i, -1, -i,$

To simplify, i^{-n} we can write $i^{-n} = \frac{1}{i^n}$ which can be simplified as above.

Example 1.1: Express the given complex number in the form $a + ib$:

(A) i^{-39}

(B) $(1 - i)^4$ [NCERT]

Ans. (A) $i^{-39} = \frac{1}{i^{39}}$

$= \frac{1}{(i^{38}) \times i}$

$= \frac{1}{(i^2)^{19} \times i}$

Putting $i^2 = -1$

$= \frac{1}{(-1)^{19} \times i}$

$= \frac{1}{-1 \times i}$

$= \frac{1}{-i}$ [Multiplying and dividing by $-i$]

$= \frac{1}{-i} \times \frac{-i}{-i}$

$= \frac{-i}{i^2}$ [Putting $i^2 = -1$]

$= \frac{-i}{(-1)} = i$

$= 0 + i$

(B) $(1 - i)^4$

$= ((1 - i)^2)^2$

$= (1 - i)^2(1 - i)^2$

Using $(a - b)^2 = a^2 + b^2 - 2ab$

$= (1^2 + i^2 - 2 \times 1 \times i)\ (1^2 + i^2 - 2 \times 1 \times i)$

$= (1 + i^2 - 2i)\ (1 + i^2 - 2i)$

$= (1 - 1 - 2i)\ (1 - 1 - 2i)$ [putting $i^2 = -1$]

$= (0 - 2i)(0 - 2i)$

$= (-2i)(-2i)$

$= 4i^2$

Putting $i^2 = -1$

$= 4 \times -1$

$= -4$

$= -4 + 0$

$= -4 + 0i$

Algebra of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then their addition, subtraction, multiplication and division are defined as

Addition of Two Complex Numbers

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

Properties of Addition:

(1) Closure: $z_1 + z_2$ is also a complex number.

(2) Commutative: $z_1 + z_2 = z_2 + z_1$

(3) Associative: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

(4) Existence of additive identity: $z + 0 = z = 0 + z$

Here, 0 is an additive identity.

(4) Existence of Additive inverse: $z + (-z) = 0 = (-z) + z$

Here, $-z$ is additive inverse.

Example 1.2: Express the following in the form of $a + ib$.

(A) $\left[\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right]-\left(-\frac{4}{3}+i\right)$

(B) $\left(\frac{1}{2}+\frac{5}{2}i\right)-\frac{3}{2}i+\left(-\frac{5}{2}-i\right)$ [NCERT]

Ans. (A) Consider the given expression,

$$\left[\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right]-\left(-\frac{4}{3}+i\right)$$

$$= \left[\left(\frac{1}{3}+4\right)+i\left(\frac{7}{3}+\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$$

$$= \left(\frac{13}{3}+\frac{8}{3}i\right)+\left(\frac{4}{3}-i\right)$$

$$= \left(\frac{13}{3}+\frac{4}{3}\right)+i\left(\frac{8}{3}-1\right)$$

$= \frac{17}{3}+\frac{5}{3}i$, which is in the form of $a + ib$.

(ii) $\left(\frac{1}{2}+\frac{5}{2}i\right)-\frac{3}{2}i+\left(-\frac{5}{2}-i\right)$

$$= \left(\frac{1}{2}-\frac{5}{2}\right)+i\left(\frac{5}{2}-\frac{3}{2}-1\right)$$

$= -2 + i0$, which is in the form of $a + ib$.

Difference of Two Complex Numbers

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

Example 1.3: Find the real values of x and y, if $(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy)$.

Ans. We have,

$$(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + 2iy)$$

$\Rightarrow \quad (x^4 + 3x^2) + (2x - y)i = 4 + (-5 + 2y)i$

On equating real and imaginary parts both sides, we get

$x^4 - 3x^2 = 4$ ...(i)

and $2x - y = -5 + 2y$

$\Rightarrow \quad 2x - 3y = -5$ ...(ii)

On solving eq. (i), we get

$x^4 - 3x^2 = 4$

$\Rightarrow \quad x^4 - 3x^2 - 4 = 0$

$\Rightarrow \quad x^4 - 4x^2 + x^2 - 4 = 0$

$\Rightarrow \quad (x^2 - 4)(x^2 + 1) = 0$

$\Rightarrow \quad x^2 - 4 = 0$

[$\because x^2 + 1 \neq 0$, for any real value of x]

$\therefore \quad x^2 = \pm 2$

On putting $x = \pm 2$ in Eq. (ii), we get

$y = 3$, when $x = 2$ and $y = \frac{1}{3}$, when $x = -2$

Thus, $x = -2, y = \frac{1}{3}$ or $x = 2, y = 3$.

Multiplication of Two Complex Numbers

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Properties of Multiplication:

(1) Closure law: z_1z_2 is also a complex number.

(2) Commutative law: $z_1z_2 = z_2z_1$

(3) Associative law: $z_1(z_2z_3) = (z_1z_2)z_3$

(4) Existence of multiplicative identity

$$z \cdot 1 = z = 1 \cdot z$$

Here, 1 is multiplicative identity.

(5) Existence of multiplicative inverse: For every non-zero complex number z, there exists a complex z_1 such that $z \cdot z_1 = 1 = z_1 \cdot z$

(6) Distributive law: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

Example 1.4: Find the real values of x and y, if $(1 + i)(x + iy) = 2 - 5i$.

Ans. We have, $(1 + i)(x + iy) = 2 - 5i$

$\Rightarrow \quad x + iy + ix + i^2y = 2 - 5i$

$\Rightarrow \quad x + i(y + x) - y = 2 - 5i \qquad [\because i^2 = -1]$

$\Rightarrow \quad (x - y) + i(x + y) = 2 - 5i$

On equating real and imaginary parts from both sides, we get

$$x - y = 2 \qquad \text{...(i)}$$

and $$x + y = -5 \qquad \text{...(ii)}$$

On adding eqs.(i) and (ii), we get

$$x - y + x + y = 2 - 5$$

$\Rightarrow \quad 2x = -3$

$\Rightarrow \quad x = \frac{-3}{2}$

On substituting $x = \frac{-3}{2}$ in eq. (ii), we get

$$\frac{-3}{2} + y = -5$$

$\Rightarrow \quad y = -5 + \frac{3}{2} = \frac{-10+3}{2} = \frac{-7}{2}$

$$x = \frac{-3}{2} \text{ and } y = \frac{-7}{2}$$

Division of Two Complex Numbers

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

Where, $z_2 \neq 0$

Example 1.5: Express $\frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$ in the form of $a + ib$. **[NCERT]**

Ans. $\frac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

$$= \frac{(3)^2 - (\sqrt{5}i)^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i}$$

$[\because (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2]$

$$= \frac{9+5}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}i} = \frac{7}{\sqrt{2}i} \times \frac{\sqrt{2}i}{\sqrt{2}i}$$

[by rationalising the denominator]

$$= \frac{7\sqrt{2}i}{2i^2} = \frac{7\sqrt{2}i}{-2} = 0 - i\frac{7\sqrt{2}}{2}$$

$$= 0 + i\left(\frac{-7\sqrt{2}}{2}\right),$$

which is in the form of $(a + ib)$.

Conjugate and Modulus

Let $z = a + ib$ be a complex number, then the conjugate of z, denoted by $\bar{z}$ is defined as the complex number

$$a - ib$$

So, $$\bar{z} = a - ib$$

In other words, the conjugate of a complex number can be obtained by changing the sign of imaginary part of z. It is denoted by $\bar{z}$.

Properties of Conjugate:

(1) $(\bar{\bar{z}}) = z$

(2) $z + \bar{z} = 2Re(z),\ z - \bar{z} = 2i\ Im(z)$

(3) $z = \bar{z}$, if z is purely real

(4) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(5) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(6) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_1}$

(7) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(8) $1\left(\frac{z_1}{z_2}\right) = \frac{\overline{z_1}}{\overline{z_1}}, \overline{z_2} \neq 0$

(9) $z . \bar{z} = \{Re(z)\}^2 + Im(z)\}^2$

(10) $z_1\overline{z_2} + \overline{z_1}z_2 = 2Re(\overline{z_1}z_2) = 2Re(z\overline{z_2})$

(11) If $z = f(z_1)$, then $\bar{z} = f(z_1)\ f(\overline{z_1})$

(12) $(\bar{z})^n = (\overline{z^n})$

Modulus

Let $z = a + ib$ be a complex number. Then the modulus of z is defined as the non-negative real number $\sqrt{a^2 + b^2}$. It is denoted by $|z|$.

So, $$|z| = \sqrt{a^2 + b^2}$$

Properties of Modulus:

(1) $|z| \geq 0$

(2) If $|z| \geq 0$, then $z = 0$ i.e. $Re(z) = 0 = Im(z)$

(3) $-|z| \leq Re(z) \leq |z|$ and $-|z| \leq |Im(z)| \leq |z|$

(4) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(5) $z.\bar{z} = |z|^2$

(6) $|z_1 z_2| = |z_1||z_2|^2$

(7) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$

(8) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re\ (z_1\overline{z_2})$

(9) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2Re\ (z_1\overline{z_2})$

(10) $|z_1 + z_2| \leq |z_1| + |z_2|$

(11) $|z_1 - z_2| \geq |z_1| - |z_2|$

Example 1.6: **What is the conjugate of $\frac{2-i}{(1-2i)^2}$?**

Ans. Let $z = \frac{2-i}{(1-2i)^2} = \frac{2-i}{(1^2 - 2(1)(2i) + (2i)^2)}$

$[\because (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2]$

$\Rightarrow \quad z = \frac{2-i}{(1-4i-4)} \qquad [\because i^2 = -1]$

$\Rightarrow \quad z = \frac{2-i}{-3-4i}$

$\Rightarrow \quad z = \frac{2-i}{-3-4i} \times \frac{-3+4i}{-3+4i}$

[by rationalising the denominator]

$\Rightarrow \quad z = \frac{(2-i)(-3+4i)}{(-3)^2 - (4i)^2}$

$[\because (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2]$

$\Rightarrow \quad z = \frac{-6+8i+3i-4i^2}{9-16i^2}$

$\Rightarrow \quad z = \frac{-6+11i+4}{9+16} \qquad [\because i^2 = -1]$

$\Rightarrow \quad z = \frac{-2+11i}{25}$

$\Rightarrow \quad z = -\frac{2}{25} + \frac{11}{25}i$

Hence, $\bar{z} = -\frac{2}{25} - \frac{11}{25}i$

Example 1.7: **Find the modulus of the complex number $\frac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}}$.**

Ans. Let $z = \frac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}}$

$= \frac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}} \times \frac{2\sqrt{3} + i\sqrt{2}}{2\sqrt{3} + i\sqrt{2}}$

[by rationalising the denominator]

$= \frac{6 + i\sqrt{6}i - 2\sqrt{6}i - 2i^2}{(2\sqrt{3})^2 - (\sqrt{2}i)^2}$

$[\because (z_1 - z_2)(z_1 + z_2) = z_1^2 - z_2^2]$

$= \frac{6 - \sqrt{6}i + 2}{12+2} \qquad [\because i^2 = -1]$

$= \frac{8 - \sqrt{6}i}{14} = \frac{8}{14} - \frac{\sqrt{6}}{14}i$

$\Rightarrow \quad z = \frac{4}{7} - \frac{\sqrt{6}}{14}i$

Now, modulus of z,

$|z| = \sqrt{\left(\frac{4}{7}\right)^2 + \left(\frac{-\sqrt{6}}{14}\right)^2}$

$= \sqrt{\frac{16}{49} + \frac{6}{196}}$

$= \sqrt{\frac{64+6}{196}} = \sqrt{\frac{70}{196}} = \sqrt{\frac{5}{14}}$

Example 1.8: **(A) Find the multiplicative inverse of the complex number $\sqrt{5} + 3i$.**

(B) Express the following expression in the form of $a + ib$.

$$\frac{(3+\sqrt{5}\,i)(3-\sqrt{5}\,i)}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-\sqrt{2}\,i)}.$$ **[NCERT]**

Ans. (A) Multiplicative inverse of $z = z^{-1}$

$= \frac{1}{z}$

Here, $z = \sqrt{5} + 3i$

Multiplicative inverse of $\sqrt{5} + 3i = \frac{1}{\sqrt{5} + 3i}$

Rationalising

$= \frac{1}{\sqrt{5}+3i} \times \frac{\sqrt{5}-3i}{\sqrt{5}-3i}$

$= \frac{\sqrt{5}-3i}{(\sqrt{5}+3i)(\sqrt{5}-3i)}$

$[\because (a-b)(a+b) = a^2 - b^2]$

$= \frac{\sqrt{5}-3i}{(\sqrt{5}+3i)(\sqrt{5}-3i)}$

$= \frac{\sqrt{5}-3i}{(\sqrt{5})^2 - (3i)^2} \qquad$ [Putting $i^2 = -1$]

$= \frac{\sqrt{5}-3i}{5-9(-1)}$

$= \frac{\sqrt{5}-3i}{5+9}$

$$= \frac{\sqrt{5}-3i}{14}$$

$$= \frac{\sqrt{5}}{14} - \frac{3}{14}i$$

(B) $\dfrac{(3+\sqrt{5}i)(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

$$= \frac{(3)^2-(i\sqrt{5})^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+i\sqrt{2}}$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \frac{9-i^2\times 5}{\sqrt{3}-\sqrt{3}+\sqrt{2}i+i\sqrt{2}}$$

Putting $i^2 = -1$

$$= \frac{9-(-1)\times 5}{0+\sqrt{2}i+i\sqrt{2}}$$

$$= \frac{14}{2\sqrt{2}i}$$

$$= \frac{14}{2\sqrt{2}i} \times \frac{2\sqrt{2}i}{2\sqrt{2}i}$$

$$= \frac{28\sqrt{2}i}{8\times i^2}$$

$$= \frac{28\sqrt{2}i}{-8}$$ [Putting $i^2 = -1$]

$$= \frac{-7\sqrt{2}}{2}i$$

$$= 0 + \left(\frac{-7\sqrt{2}}{2}\right)i$$

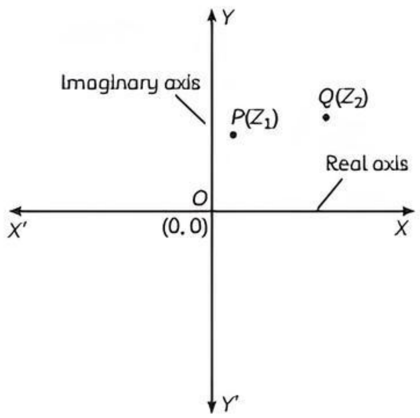
TOPIC 2

ARGAND PLANE

A complex number $z = x + iy$ can be represented by a unique point $P(x, y)$ in the Cartesian plane, referred to a pair of rectangular axes. The representation of complex numbers as points in the plane is known as argand diagram. The plane, representing complex numbers as points, is called complex plane or argand plane or Gaussian plane.

A purely real number x, *i.e.* $(x + 0i)$ is represented by the point $(x, 0)$ on *x-axis*. Therefore, *x-axis* is called the real axis, while a is the intersection (common) of two axes called zero complex number *i.e.* $z = 0 + 0i$.

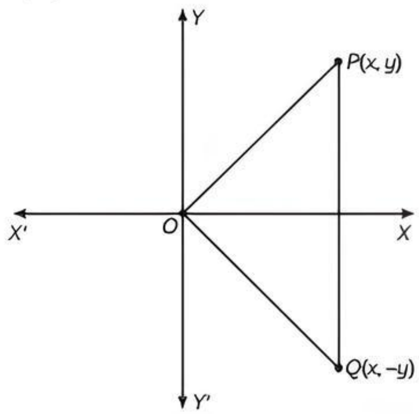
If two complex numbers Z_1 and Z_2 are represented by the points P and Q in the complex plane, then $|Z_1 - Z_2| = PQ$ = distance between P and Q.



Representation of Complex Number on Argand Plane

The complex number $z = x + iy$ and its conjugate $z = x - iy$ can be represented in the Argand plane. The complex number and conjugate are the points $P(x, y)$ and $Q(x, -y)$ respectively.

Geometrically, the point $(x, -y)$ is the mirror image of the point (x, y) on the real axis.



A complex number satisfying $-\pi < \theta \leq \pi$ is called Principal argument denoted by α.

Properties of Argand Plane

The qualities of the argand plane listed below can help you better comprehend the argand plane.

(1) The axes of the argand plane are comparable to the axes of ordinary coordinates.

(2) The origin is the place where the real and imaginary axes of the argand plane intersect.

(3) The argand plane's real and imaginary axes are perpendicular to one other.

(4) The real and imaginary axes of the argand plane split it into four quadrants in the same way as the coordinate axis does.

(5) The distance and midpoint formulas are the same in the argand plane as they are in the coordinate axes.

(6) Cartesian coordinates or polar coordinates are used to represent the points in the argand plane.

Example 1.9: Find the modulus and the argument of the complex number $z = -\sqrt{3} + i$. **[NCERT]**

Ans. Let, $z = x + iy$

Modulus of $z = |z| = \sqrt{x^2 + y^2}$

Method (1): To calculating modulus of z

Given, $z = -\sqrt{3} + i$

Complex number z is of the form $x + iy$.

Where, $x = -\sqrt{3}$ and $y = 1$

$$\text{Modulus of } z = |z| = \sqrt{x^2+y^2} = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = \sqrt{4}$$

Or, $|z| = 2$

Hence, Modulus of $z = 2$

Method (2)

Given, $z = -\sqrt{3} + i$ ...(i)

Let, $z = r(\cos\theta + i\sin\theta)$ ...(ii)

Here, r is modulus, and θ is argument

From (i) and (ii), we get

$$-\sqrt{3} + i = r(\cos\theta + i\sin\theta)$$

$$-\sqrt{3} + i = r\cos\theta + ir\sin\theta$$

$-\sqrt{3}$ → Real part; i → Imaginary part; $r\cos\theta$ → Real Part; $ir\sin\theta$ → Imaginary part

On comparing real parts, we get

$$-\sqrt{3} = r\cos\theta$$

On squaring both sides, we get

$$(\sqrt{3})^2 = (r\cos\theta)^2$$

$$3 = r^2\cos^2\theta \quad \text{...(iii)}$$

Similarly, comparing imaginary parts

$$1 = r\sin\theta$$

On squaring both sides, we get

$$(1)^2 = (r\sin\theta)^2$$

$$1 = r^2\sin^2\theta \quad \text{...(iv)}$$

Adding (iii) and (iv), we get

$$3 + 1 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$4 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$4 = r^2(\cos^2\theta + \sin^2\theta)$$

$$4 = r^2 \times 1 \quad [\cos^2\theta + \sin^2\theta = 1]$$

$$4 = r^2$$

$$r = 2$$

Hence, modulus = 2

Finding argument

$$-\sqrt{3} + i = r\cos\theta + ir\sin\theta$$

$-\sqrt{3}$ → Real part; i → Imaginary part; $r\cos\theta$ → Real Part; $ir\sin\theta$ → Imaginary part

Comparing real part

$$-\sqrt{3} = r\cos\theta$$

Putting $r = 2$

$$-\sqrt{3} = 2\cos\theta$$

$$-\sqrt{3} = 2\cos\theta$$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

On comparing imaginary parts, we get

$$1 = r\sin\theta$$

Putting, $r = 2$

$$1 = 2 \times \sin\theta$$

$$\frac{1}{2} = \sin\theta$$

$$\sin\theta = \frac{1}{2}$$

Hence, $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$

Since $\sin\theta$ is positive and $\cos\theta$ is negative,

Argument will be in IInd quadrant

$$\text{Argument} = 180° - 30° = 150° = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$$

Example 1.10: If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$.

Ans. We know that, $|z|^2 = (z)(\bar{z})$

$z = x + iy$, $\bar{z}$ = conjugate

$\bar{z} = x - iy$, $|z|$ = modulus, $|z| = \sqrt{x^2 + y^2}$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\overline{\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)}$$

$$= \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\overline{\beta-\alpha}}{\overline{1-\bar{\alpha}\beta}}\right)$$

$$= \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\bar{\bar{\alpha}}\bar{\beta}}\right)$$

$$= \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right)$$

$$= \frac{(\beta-\alpha)(\bar{\beta}-\alpha)}{(1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})}$$

[$\because (\bar{\bar{\alpha}} = \alpha)$ and Conjugate of 1 is 1 i.e. $\bar{1} = 1$]

$$= \frac{\beta(\bar{\beta}-\bar{\alpha})-\alpha(\bar{\beta}-\bar{\alpha})}{1(1-\alpha\bar{\beta})-\bar{\alpha}\beta(1-\alpha\bar{\beta})}$$

$$= \frac{\beta\bar{\beta}-\beta\bar{\alpha}-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\alpha\bar{\alpha}\beta\bar{\beta}}$$

As $|z| = (z)(\bar{z})$

$$= \frac{|\beta|^2-\beta\bar{\alpha}-\bar{\beta}\alpha+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2|\beta|^2}$$

Given that $|\beta| = 1$

So, $|\beta|^2 = 1$

$$= \frac{1-\beta\bar{\alpha}-\bar{\beta}\alpha+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2 1}$$

$$= \frac{1-\beta\bar{\alpha}-\bar{\beta}\alpha+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}$$

$$= 1$$

Hence, $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = 1$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \sqrt{1}$$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1$$

Example 1.11: Express the given complex number in the form $a + ib$.

$$\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$$ [NCERT]

Ans. Let $z = \left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$

$$= \frac{1}{5}+i\frac{2}{5}-4-i\frac{5}{2}$$

$$= \frac{1}{5}-4+i\frac{2}{5}-i\frac{5}{2}$$

$$= \left(\frac{1}{5}-4\right)+\left(i\frac{2}{5}-i\frac{5}{2}\right)$$

$$= \left(\frac{1-4\times5}{5}\right)+i\left(\frac{2\times2-5\times5}{10}\right)$$

$$= \left(\frac{1-20}{5}\right)+i\left(\frac{4-25}{10}\right)$$

$$= \left(\frac{-19}{5}\right)+\left(\frac{-21}{10}i\right)$$

which is in the form of $a + ib$.

Example 1.12: Express the given complex number form $a + i\,b$.

$$\left(-2-\frac{1}{3}i\right)^3$$ [NCERT]

Ans. Let $z = \left(-2-\frac{1}{3}i\right)^3$

$$= -1\left(2+\frac{1}{3}i\right)^3$$

It is of the form $(a + b)^3$

[$\because$ Using $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$]

Here, $a = 2$ and $b = \frac{1}{3}i$

$$z = -1\left((2)^3+\left(\frac{1}{3}i\right)^3+3\times2\times\frac{1}{3}i\left(2+\frac{1}{3}i\right)\right)$$

$$= -1\left(8+\left(\frac{1}{3}\right)^3\times(i)^3+2i\left(2+\frac{1}{3}i\right)\right)$$

$$= -1\left(8+\frac{1}{27}i^3+4i+\frac{2}{3}i^2\right)$$

$$= -1\left(8+\frac{1}{27}i\times i^2+4i+\frac{2}{3}i^2\right)$$

Putting $i^2 = -1$, we get

$$z = -1\left(8+\frac{1}{27}i(-1)+4i+\frac{2}{3}(-1)\right)$$

$$z = -1\left(8-\frac{1}{27}i+4i-\frac{2}{3}\right)$$

$$z = -8+\frac{1}{27}i-4i+\frac{2}{3}$$

$$= -8+\frac{2}{3}+\frac{1}{27}i-4i$$

$$= \left(-8+\frac{2}{3}\right)+\left(\frac{1}{27}-4\right)i$$

$$= \left(\frac{-24+2}{3}\right)+\left(\frac{1-4\times 27}{27}\right)i$$

$$= \frac{-22}{3}+\left(\frac{1-108}{27}\right)i$$

$$= \frac{-22}{3}+\left(\frac{-107}{27}\right)i$$

$$= \frac{-22}{3}-\frac{107}{27}i$$

Which is in the form of $a + ib$.

Example 1.13: Solve the equations: $x^2 - x + 2 = 0$.

Ans. The given equation is $x^2 - x + 2 = 0$

which is of the form

$$ax^2 + bx + c = 0$$

Comparing $a = 1, b = -1, c = 2$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(-1)\pm\sqrt{(-1)^2-4\times 1\times 2}}{2\times 1}$$

$$= \frac{1\pm\sqrt{1-8}}{2}$$

$$= \frac{1\pm\sqrt{-7}}{2}$$

$$= \frac{1\pm\sqrt{-1\times 7}}{2}$$

$$= \frac{1\pm\sqrt{-1}\times\sqrt{7}}{2}$$

$$= \frac{1\pm\sqrt{7}i}{2}$$

Thus, $x = \frac{1\pm\sqrt{7}i}{2}$

Example 1.14: If $a + ib = \frac{(x-i)^2}{2x^2+1}$ prove that $a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$. [NCERT]

Ans. Given that

$$a+ib = \frac{(x-i)^2}{2x^2+1}$$

$$\Rightarrow |a+ib| = \left|\frac{(x-i)^2}{2x^2+1}\right|$$

[On taking modulus on both sides]

$$\Rightarrow |a+ib| = \left|\frac{(x-i)(x-i)}{2x^2+1}\right|$$

$$\Rightarrow |a+ib| = \frac{|(x-i)(x-i)|}{|2x^2+1|}$$

$$\Rightarrow |a+ib| = \frac{|x-i||x-i|}{2x^2+1}$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{\sqrt{x^2+1}\sqrt{x^2+1}}{2x^2+1}$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{x^2+1}{2x^2+1}$$

$$\Rightarrow a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

[on squaring both sides]

Hence, proved.

Example 1.15: Case Based:

The conjugate of a complex number z, is the complex number, obtained by changing the sign of imaginary part of z. It is denoted by $\bar{z}$. The modulus (or absolute value) of a complex number, $z = a + ib$ is defined as the non-negative real number $\sqrt{a^2+b^2}$. It is denoted by $|z|$. i.e., $|z| = \sqrt{a^2+b^2}$.

Multiple inverse of z is $\frac{z}{|z^2|}$. It is also called reciprocal of z.

$$2\bar{z} = |z^2|$$

On the basis of above information, answer the following questions:

(A) Assertion (A): If $(x - iy)(3 + 5i)$ is the conjugate of $(-6 - 24i)$, then the value of $x + y$ is 1.

Reason (R): The conjugate of $(-3 -4i)$ is $(-3 + 4i)$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).
(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).
(c) (A) is true but (R) is false.
(d) (A) is false but (R) is true.

(B) The value of $(z+3)(\bar{z}+3)$ is equivalent to:

(a) $|z + 3|^2$ (b) $|z - 3|$
(c) $z^2 + 3$ (d) none of these

(C) If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is:

(a) $\frac{|z|}{2}$ (b) $|z|$
(c) $2|z|$ (d) none of these

(D) If $z_1 = 1 - 3i$ and $z_2 = -2 + 4i$, then find $|z_1 + z_2|$.

(E) If $z = 3 + 4i$, then find $\frac{z+\bar{z}}{2}$.

Ans. (A) *(d)* *(A) is false but (R) is true*

Explanation: We have, $(x - iy).(3 + 5i)$ is the conjugate of $-6 - 24i$.

$\Rightarrow \quad (x - iy)(3 + 5i) = -6 + 24i$

$[\because$ conjugate of $-6 - 24i = -6 + 24i]$

$\Rightarrow \quad 3x - 3iy + 5ix - 5i^2y = -6 + 24i$

$\Rightarrow \quad (3x + 5y) + i(5x - 3y) = -6 + 24i$

$[\because i^2 = -1]$ ...(i)

On equating real and imaginary parts both sides of eq. (i), we get

$3x + 5y = -6$ ...(ii)

and $5x - 3y = 24$ ...(iii)

On multiplying eq. (i) by 3 and eq. (ii) by 5, then adding the result, we get

$9y + 15y + 25x - 15y = -18 + 120$

$\Rightarrow \quad 34x = 102$

$\Rightarrow \quad x = 3$

On substituting $x = 3$ in eq. (ii), we get

$9 + 5y = -6$

$\Rightarrow \quad 5y = -15$

$\Rightarrow \quad y = -3$

$\Rightarrow \quad x + y = 3 + (-3) = 0$

(B) *(a)* $|z + 3|^2$

Explanation: Given that, $(z+3)(\bar{z}+3)$

Let $z = x + iy$

$\Rightarrow \quad (z+3)(\bar{z}+3) = (x + iy + 3)(x + 3 - iy)$

$= (x + 3)^2 - (iy)^2$

$= (x + 3)^2 + y^2$

$= |x + 3 + iy|^2$

$= |z + 3|^2$

(C) *(a)* $\frac{|z|}{2}$

Explanation:

Let $z = 1 + 2i$

$\Rightarrow \quad |z| = \sqrt{1+4} = \sqrt{5}$

Now, $f(z) = \frac{7-z}{1-z^2} = \frac{7-1-2i}{1-(1+2i)^2}$

$= \frac{6-2i}{1-1-4i^2-4i} = \frac{6-2i}{4-4i}$

$= \frac{3-i}{2-2i} = \frac{(3-i)(2+2i)}{(2-2i)(2+2i)}$

$= \frac{6-2i+6i-2i^2}{4-4i^2}$

$= \frac{6+4i+2}{4+4}$

$= \frac{8+4i}{8} = 1 + \frac{1}{2}i$

$f(z) = 1 + \frac{1}{2}i$

$\therefore \quad |f(z)| = \sqrt{1+\frac{1}{4}} = \sqrt{\frac{4+1}{4}}$

$= \frac{\sqrt{5}}{2} = \frac{|z|}{2}$

(D) Given, $z_1 = 1 - 3i$ and $z_2 = -2 + 4i$

$\therefore \quad z_1 + z_2 = (1 - 3i) + (-2 + 4i) = -1 + i$

$|z_1 + z_2| = \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

(E) Given, $z = 3 + 4i$

$\therefore \quad \bar{z} = 3 - 4i$

$\Rightarrow \quad z + \bar{z} = (3+4i) + (3-4i) = 6$

Now, $\frac{z+\bar{z}}{2} = \frac{6}{2} = 3$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. If arg $(z - 1)$ = arg $(z - 3i)$, then $(x - 1) : y$ is:

(a) 3 : 1 (b) 1 : 3

(c) 3 : 2 (d) 2 : 3 [Diksha]

Ans. *(b) 1 : 3*

Explanation: Given arg $(z - 1)$ = arg $(z + 3i)$

$\arg(x + iy - 1) = \arg(x + iy + 3i)$

$\arg(x - i + iy) = \arg(x + i(y + 3))$

$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x-1}\right) = \tan^{-1}\left(\frac{y+3}{x}\right)$

$\therefore \quad \frac{y}{x-1} = \frac{y+3}{x}$

$\Rightarrow \quad xy = xy - y + 3x - 3$

$\Rightarrow \quad y = 3(x - 1)$

$\Rightarrow \quad \frac{1}{3} = \frac{x-1}{y}$

$\therefore \quad x - 1 : y = 1 : 3$

2. If $a + ib = \frac{(p^2+1)^2}{2p-i}$ then $a^2 + b^2$ is:

(a) $\frac{(p^2+1)^4}{4p^2+1}$ (b) $\frac{(p+1)^2}{4p+1}$

(c) $\frac{(p-1)^2}{(4p-1)^2}$ (d) none of these

Ans. (a) $\frac{(p^2+1)^4}{4p^2+1}$

Explanation: Given, $a + ib = \frac{(p^2+1)^2}{2p-i}$ ...(i)

Then, $a - ib = \frac{(p^2+1)^2}{2p+i}$ ...(ii)

On multiplying (i) and (ii), we get

$$a^2+b^2 = \frac{(p^2+1)^2}{2p-i} \times \frac{(p^2+1)^2}{2p+i} = \frac{(p^2+1)^4}{4p^2+1}$$

3. If $(a + ib)^{1/3} = p + iq$ then $\frac{a}{p} + \frac{b}{q}$ is:

(a) 0 (b) 1

(c) –1 (d) none of these

Ans. (d) *none of these*

Explanation: Given, $(a + ib)^{1/3} = p + iq$

It can be rewrite as $a + ib = (p + iq)^3$

$$a + ib = p^3 + (iq)^3 + 3p^2iq + 3p(iq)^2$$
$$= p^3 - iq^3 + 3p^2qi - 3pq^2$$
$$a + ib = (p^3 - 3pq^2) + i\,(3p^2q - q^3)$$

Thus, $a = p^3 - 3pq^2$ and $b = 3p^2q - q^3$

So, $$\frac{a}{p}+\frac{b}{q} = \frac{p(p^2-3q^2)}{p} + \frac{q(3p^2-q^2)}{q}$$
$$= p^2 - 3q^2 + 3p^2 - q^2$$
$$= 4\,(p^2 - q^2)$$

4. If $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is:

(a) $\frac{|z|}{2}$ (b) $|z|$

(c) $2|z|$ (d) none of these

[Delhi Gov. SQP 2022]

Ans. (a) $\frac{|z|}{2}$

Explanation: Given, $f(z) = \frac{7-z}{1-z^2}$, where $z = 1 + 2i$

$\Rightarrow \quad |z| = \sqrt{1^2+2^2} = \sqrt{5}$

$\Rightarrow \quad f(z) = \frac{7-z}{1-z^2}$

$$= \frac{7-(1+2i)}{1-(1+2i)^2}$$
$$= \frac{7-1-2i}{1-1-4i^2-4i}$$
$$= \frac{6-2i}{4-4i}$$
$$= \frac{3-i}{2-2i}$$
$$= \frac{3-i}{2-2i} \times \frac{2+2i}{2+2i}$$
$$= \frac{6+6i-2i-2i^2}{4-4i}$$
$$= \frac{6+4i+2}{4+4}$$
$$= \frac{8+4i}{8}$$
$$= 1+\frac{1}{2}i$$

$\Rightarrow \quad |f(z)| = \sqrt{1^2+\left(\frac{1}{2}\right)^2} = \sqrt{1+\frac{1}{4}}$

$$= \frac{\sqrt{5}}{2} \Rightarrow \frac{|z|}{2}$$

5. $\left(\sqrt{-4}\right)\left(\sqrt{-2}\right)$ is equal to:

(a) $\sqrt{8}$ (b) $-\sqrt{8}$

(c) $i\sqrt{8}$ (d) none of these

Ans. (b) $-\sqrt{8}$

Explanation: $(\sqrt{-4})(-\sqrt{2}) = i\sqrt{4}.i\sqrt{2}$

$$= i^2\sqrt{8} = -\sqrt{8}$$

6. If $a + ib = 3 - 4i$ then *Re* and *Im* part of the complex number are:

(a) 3, 4 (b) 3, –4

(c) 4, 3 (d) 4, –3

Ans. (b) 3, –4

Explanation: Here, complex number is

$$a + ib = 3 - 4i$$

A general complex number can be written as Re + i(Im).

So, Re = 3

And Im = –4

7. $1 + i^2 + i^4 + i^6 + ... + i^{2n}$ is:

(a) positive

(b) negative

(c) 0

(d) cannot be determined

[Delhi Gov. Term-1 SQP 2021]

Ans. *(d) cannot be determined*

Explanation: $S = 1 + i^2 + i^4 + i^6 + ... + i^{2n}$

$= 1 - 1 + 1 - 1 + ... + (-1)^n$

Obviously, it depends on n.

Hence cannot be determined unless n is known.

8. If $\frac{1-in}{1+in} = a + ib$ then $a^2 + b^2 =$

(a) 1 (b) –1

(c) 0 (d) none of these

Ans. *(a) 1*

Explanation: If $\frac{1-in}{1+in} = a+ib$ ...(i)

then

$$\frac{1+in}{1-in} = a-ib \quad ...(ii)$$

On multiplying eq. (i) and (ii), we get

$$(a+ib)(a-ib) = \left(\frac{1-in}{1+in}\right)\times\left(\frac{1+in}{1-in}\right)$$

$$a^2+b^2 = \left(\frac{1-in}{1+in}\right)\left(\frac{1+in}{1-in}\right) = 1$$

Hence, $a^2 + b^2 = 1$.

9. The value of $(1 + i)^4 - (1 - i)^4$ is:

(a) 8 (b) 4

(c) –8 (d) –4

Ans. *(c) –8*

Explanation: $(1 + i)^4 - (1 - i)^4$

$= ((1 + i)^2)^2 - ((1 - i)^2)^2$

$= (i^2 + 1 + 2i)^2 - (1 + i^2 - 2i)^2$

$= (-1 + 1 + 2i)^2 - (1 - 1 - 2i)^2$

$= (2i)^2 - (-2i)^2$

$= 4i^2 + 4i^2$

$= 8i^2 = -8$

10. The real value of θ for which the expression $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number is:

(a) $n\pi \pm \frac{\pi}{4}$ (b) $n\pi \pm \frac{\pi}{2}$

(c) $n\pi \pm \frac{\pi}{2}$ (d) $2n\pi \pm \frac{\pi}{4}$ [Diksha]

Ans. (c) $n\pi \pm \frac{\pi}{2}$

Explanation: Let

$$z = \frac{1+i\cos\theta}{1-2i\cos\theta}$$

$$= \frac{(1+i\cos\theta)(1+2i\cos\theta)}{(1-2i\cos\theta)(1+2i\cos\theta)}$$

$$= \frac{1-2\cos^2\theta + i\,3\cos\theta}{1+4\cos^2\theta}$$

$$= \frac{1-2\cos^2\theta}{1+4\cos^2\theta} + i\left(\frac{3\cos\theta}{1+4\cos^2\theta}\right)$$

It is given that z is purely real, then

$Im(z) = 0$

$3\cos\theta = 0$

$$\theta = \frac{(2n\pm1)\pi}{2} \text{ where } n \in N$$

$$\theta = n\pi \pm \frac{\pi}{2}$$

11. The real part of $\frac{(1+i)^2}{(3-i)}$ is:

(a) $\frac{1}{3}$ (b) $\frac{1}{5}$

(c) $-\frac{1}{3}$ (d) none of these

[Delhi Gov. Term-1 SQP 2021]

Ans. *(d) none of these*

Explanation: Given,

$$\frac{(1+i)^2}{(3-i)} = \frac{2i}{3-i} \quad [\text{Since } (i+1)^2 = 2i]$$

$$= \frac{2i(3+i)}{3^2 - i^2}$$

[After rationalising the denominator]

$$= \frac{2i(3+i)}{10}$$

$$= \frac{(3i-1)}{5}.$$

Now real part of the given complex number is $-\frac{1}{5}$.

12. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then which of the following is correct?

(a) $x = 2n$ (b) $x = 2n + 1$

(c) $x = 4n$ (d) $x = 4n + 1$ [Diksha]

Ans. *(c)* $x = 4n$

Explanation: Given $\left(\frac{1+i}{1-i}\right)^x = 1$

On multiplying and dividing with $(1 + i)^x$, we get

$$1 = \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x$$

$$1 = \left(\frac{1-1+2i}{2}\right)^x$$

$$1 = i^x$$

But $\quad i^x = 1 = i^4,$

or $\quad i^x = 1^{4n}\ (n \in z)$

Therefore, $x = 4n,\ n \in z$

13. The real value of α for which the expression $\frac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely real is:

(a) $(n+1)\frac{\pi}{2}$ (b) $(2n+1)\frac{\pi}{2}$

(c) $n\pi$ (d) none of these

[NCERT Exemplar]

Ans. *(c)* $n\pi$

Explanation: Given, $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$

Rationalising the denominator

$$\Rightarrow \frac{(1-i\sin\alpha)(1-2i\sin\alpha)}{(1+2i\sin\alpha)(1-2i\sin\alpha)}$$

$$= \frac{1-i\sin\alpha-2i\sin\alpha+2i^2\sin^2\alpha}{1-4i^2\sin^2\alpha}$$

$$= \frac{1-3i\sin\alpha-2\sin^2\alpha}{1+4\sin^2\alpha}$$

$$= \frac{1-2\sin^2\alpha}{1+4\sin^2\alpha} - \frac{3i\sin\alpha}{1+4\sin^2\alpha}$$

It is given that z is purely real, then

$$\text{Im}\,|z| = 0$$

$$\Rightarrow \frac{-3i\sin\alpha}{1+4\sin^2\alpha} = 0$$

$$\Rightarrow -3\sin\alpha = 0$$

$$\Rightarrow \sin\alpha = 0$$

$$\alpha = n\pi, n \in 1$$

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R).

Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

14. Assertion (A): Simplest form of i^{35} is $-i$.

Reason (R): Additive inverse of $(1 - i)$ is equal to $-1 + i$.

Ans. *(A) is false but (R) is true.*

Explanation: $i^{35} = \frac{1}{i^{-35}} = \frac{1}{(i^2)^{17}i}$

$$= \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = i$$

$\therefore$ Additive inverse of z is $-z$.

So, additive inverse of $(1 - i)$ is $-1 + i$.

15. Assertion (A): If $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$ then $z_1 - z_2 = -1 + 5i$.

Reason(R): If $z_1 = (a + ib)$ and $z_2 = (c + id)$ then $z_1 - z_2 = (a - c) + i(b - d)$.

Ans. *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: Given $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$ then

We know that, if $z_1 = (a + ib)$ and $z_2 = (c + id)$, then,

$$z_1 - z_2 = (a - c) + i(b - d)$$
$$z_1 - z_2 = (2 + 3i) - (3 - 2i)$$
$$z_1 - z_2 = 2 + 3i - 3 + 2i$$
$$z_1 - z_2 = -1 + 5i$$

16. Assertion (A): If $(1 + i)^6 = a + ib$ then $b = -8$.

Reason (R): If $(1 - i)^3 = a + ib$ then $\frac{a}{b} = 1$.

Ans. *(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).*

Explanation: Given,

$$(1+i)^6 = ((1+i)^2)^3$$
$$= (1 + i^2 + 2i)^3$$
$$= (1 - 1 + 2i)^3$$
$$= (2i)^3$$
$$= 8i^3$$
$$= 8i \times i^2$$
$$= -8i$$
$$\therefore \quad z = 0 - 8i$$

On comparing it with $a + ib$, we get

$$b = -8$$

Given, $(1 - i)^3 = 1^3 - i^3 - 3 \times 1 \times i(1 - i)$

$= 1 - i^2 \times i - 3i + 3i^2$

$= 1 + i - 3i - 3$

$= -2 - 2i$

$\therefore \quad z = -2 - 2i$

On comparing it with $a + ib$, we get

$a = -2$ and $b = -2$

$\therefore \quad \frac{a}{b} = \frac{-2}{-2} = 1$

17. Assertion (A): Multiplicative inverse of $2 - 3i$ is $2 + 3i$.

Reason (R): If $z = 3 + 4i$ then $\bar{z} = 3 - 4i$.

Ans. *(A) is false but (R) is true.*

Explanation: Multiplicative inverse of $z = z^{-1}$

Multiplicative inverse of $z = \frac{1}{z}$

Putting $z = 2 - 3i$

Multiplicative inverse of $2 - 3i = \frac{1}{2-3i}$

$$= \frac{1}{(2-3i)} \times \frac{(2+3i)}{(2+3i)}$$

$$= \frac{2+3i}{(2-3i)(2+3i)}$$

[Using $(a + b)(a - b) = a^2 - b^2$]

$$= \frac{2+3i}{2^2-(3i)^2}$$

$$= \frac{2+3i}{4-9i^2}$$

Putting $i^2 = -1$

$$= \frac{2+3i}{4-9\times-1}$$

$$= \frac{2+3i}{4+9}$$

$$= \frac{2+3i}{13}$$

$$= \frac{2}{13}+\frac{3i}{13}$$

Hence, Multiplicative inverse $= \frac{2}{13}+\frac{3i}{13}$.

$z = a + ib$

$\bar{z} = a - ib$

$z = 3 + 4i$

then $\bar{z} = 3 - 4i$

CASE BASED Questions (CBQs)

[**4** & **5** marks]

Read the following passages and answer the questions that follow:

18. *Two complex numbers $Z_1 = a + ib$ and $Z_2 = c + id$ are said to be equal if $a = c$ and $b = d$.*

(A) If $(2a + 2b) + i(b - a) = -4i$, then find the real values of a and b.

(B) If $(x + y) + i(x - y) = 4 + 6i$, then find xy.

(C) Express the given expression $(1 + i)(1 + 2i)$ in the form $a + ib$ and find the values of a and b.

Ans. (A) We have $(2a + 2b) + i(b - a) = -4i$

Here

$2a + 2b = 0$

$\Rightarrow \quad a + b = 0 \quad$ ...(i)

and $\quad b - a = -4 \quad$ ...(ii)

On adding eq. (i) and (ii), we get

$a = 2$ and $b = -2$

(B) Given that $(x + y) + i(x - y) = 4 + 6i$

Hence,

$x + y = 4$

$x - y = 6$

by solving equations

$x = 5, y = -1$

then $\quad xy = -5$

(C) Given expression: $(1 + i)(1 + 2i)$

Hence,

$(1 + i)(1 + 2i) = 1(1) + 1(2i) + i + 2i(i)$

$(1 + i)(1 + 2i) = 1 + 2i + i + 2i^2$

$(1 + i)(1 + 2i) = 1 + 2i + i + 2(-1)$ [As, $i^2 = -1$]

$(1 + i)(1 + 2i) = 1 + 2i + i - 2$

$(1 + i)(1 + 2i) = -1 + 3i$

Hence, the expression $(1 + i)(1 + 2i)$ in the form of $a + bi$ is $-1 + 3i$.

Thus, the value of $a = -1$ and $b = 3$.

19. *A complex number z is pure real if and only if $\bar{z} = z$ and is pure imaginary if and only if $\bar{z} = -z$.*

(A) If $(1 + i)z = (1 - i)\bar{z}$, then $-i\bar{z}$ is:

(a) $-\bar{z}$ (b) z

(c) $\bar{z}$ (d) z^{-1}

(B) $\overline{z_1 z_2}$ is:

(a) $\overline{z_1}\,\overline{z_2}$ (b) $\overline{z_1} + \overline{z_1}$

(c) $\dfrac{\overline{z_1}}{\overline{z_2}}$ (d) $\dfrac{1}{\overline{z_1 z_2}}$

(C) If x and y are real numbers and the complex number

$$\frac{(2+i)x-i}{4+i}+\frac{(1-i)y+2i}{4i}$$

is pure real, the relation between x and y is:

(a) $8x - 17y = 16$
(b) $8x + 17y = 16$
(c) $17x - 8y = 16$
(d) $17x - 8y = -16$

(D) If $z = \dfrac{3+2i\sin\theta}{1-2i\sin\theta}\left(0<\theta\le\dfrac{\pi}{2}\right)$ is pure imaginary, then θ is equal to:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{12}$

(E) Assertion (A): The value of the expression: $i^{30} + i^{40} + i^{60}$ is 1.

Reason (R): The values of i^4 and i^2 are 1 and –1.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).
(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).
(c) (A) is true but (R) is false.
(d) (A) is false but (R) is true.

Ans. (A) *(b)* z

Explanation: Since, $(1 + i)z = (1 - i)\,\overline{z}$

$$\Rightarrow \quad \frac{z}{\overline{z}}=\frac{1-i}{1+i}\times\frac{1-i}{1-i}=\frac{(1-i)^2}{1-i^2}$$

$$=\frac{1+i^2-2i}{1+1}=-i$$

$$\Rightarrow \quad z=-i\,\overline{z}$$

(B) *(a)* $\overline{z_1}\,\overline{z_2}$

Explanation: $\because \overline{z_1 z_2}=\overline{z_1}\,\overline{z_2}$

(C) *(a)* $8x - 17y = 16$

Explanation: Let

$$z=\frac{(2+i)x-i}{4+i}+\frac{(1-i)y+2i}{4i}$$

$$=\frac{2x+(x-1)i}{4+i}+\frac{y+(2-y)i}{4i}\times\frac{i}{i}$$

$$=\frac{(2x+(x-1)i)(4-i)}{(4+i)(4-i)}+\frac{-iy+(2-y)}{4}$$

$$=\frac{8x+x-1+i(4x-4-2x)}{16+1}+\frac{(2-y)-iy}{4}$$

$$=\frac{9x-1+i(2x-4)}{17}+\frac{2-y-iy}{4}$$

Since, z is real

$$\Rightarrow \quad \overline{z}=z$$

$$\Rightarrow \quad Im\,z=0$$

$$\Rightarrow \quad \frac{2x-4}{17}-\frac{y}{4}=0$$

$$\Rightarrow \quad 8x-16=17y$$

$$\Rightarrow \quad 8x-17y=16$$

(D) *(c)* $\frac{\pi}{3}$

Explanation: $z=\dfrac{3+2i\sin\theta}{1-2i\sin\theta}\times\dfrac{(1+2i\sin\theta)}{(1+2i\sin\theta)}$

$$=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta}$$

$$=\frac{(3-4\sin^2\theta)+i(8\sin\theta)}{1+4\sin^2\theta}$$

Since, z is pure imaginary.

$$\Rightarrow \quad Re(z)=0$$

$$\Rightarrow \quad \frac{3-4\sin^2\theta}{1+4\sin^2\theta}=0$$

$$\Rightarrow \quad \sin^2\theta=\frac{3}{4}$$

$$\Rightarrow \quad \sin\theta=\pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \theta=\frac{\pi}{3}\left(\text{since, }0<\theta\le\frac{\pi}{2}\right)$$

(E) *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: $i^{30} + i^{40} + i^{60}$.

The given expression can be simplified as follows:

$i^{30} + i^{40} + i^{60} = (i^4)^7 i^2 + (i^4)^{10} + (i^4)^{15}$

We know that the value of i^4 is 1.

$i^{30} + i^{40} + i^{60} = (1)^7 i^2 + (1)^{10} + (1)^{15}$

$i^{30} + i^{40} + i^{60} = (1)i^2 + 1 + 1$

$i^{30} + i^{40} + i^{60} = -1 + 1 + 1$ [Since $i^2 = 1$]

$i^{30} + i^{40} + i^{60} = 1$

Therefore, the simplification of $i^{30} + i^{40} + i^{60}$ is 1.

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

20. Reduce $\frac{(4+i\sqrt{3})(4-i\sqrt{3})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$ to the standard form.

Ans. $\frac{(4+i\sqrt{3})(4-i\sqrt{3})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}=\frac{16-3i^2}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i}$

$=\frac{16+3}{2\sqrt{2}i}=\frac{19}{2\sqrt{2}i}$

$=\frac{19}{2\sqrt{2}i}\times\frac{2\sqrt{2}i}{2\sqrt{2}i}$

$=\frac{38\sqrt{2}i}{8i^2}=\frac{-19\sqrt{2}i}{4}.$

21. Write in the form of $a + ib$: $\frac{1}{-2+\sqrt{-3}}$

[Delhi Gov. QB 2022]

Ans. Let $z=\frac{1}{-2+\sqrt{-3}}$

$=\frac{1}{-2+i\sqrt{-3}}$

$=\left(\frac{-2-i\sqrt{3}}{(-2+i\sqrt{-3})(-2-i\sqrt{3})}\right)$

$=\frac{-2-\sqrt{3}i}{4+3}$

$=\frac{-2}{7}-\frac{\sqrt{3}}{7}i$

22. Write the additive inverse of $6i-i\sqrt{-49}$.

Ans. Given, $6i-i\sqrt{-49}$

$= 6i - i\,(7i)$

$= 6i - 7i^2$

$= 6i + 7$

Additive inverse $= -6i - 7$

23. Write the value of $i + i^{10} + i^{20} + i^{30}$.

[Delhi Gov. SQP 2022]

Ans. $i + i^{10} + i^{20} + i^{30} = i + (i^2)^5 + (i^2)^{10} + (i^2)^{15}$

$= -i + (-1)^5 + (-1)^{10} + (-1)^{15}$

$= i - 1 + 1 - 1$ $\quad [\because\ i^2 = -1]$

$= i - 1$

24. Using conjugate and modulus of a complex number $z = 4 - 3i$, find the multiplicative inverse of $4 - 3i$.

Ans. Let $z = 4 - 3i$

Then, $\bar{z} = 4 + 3i$ and $|z| = \sqrt{(4)^3+(-3)^3} = 5$

Now, $z\bar{z} = |z|^2$

$\Rightarrow \frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{4+3i}{(5)^2}=\frac{4}{25}+\frac{3}{25}i.$

Hence, the multiplicative inverse of

$4-3i$ is $\frac{4}{25}+\frac{3}{25}i.$

25. If $(1 + i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

[NCERT Exemplar]

Ans. We have,

$(1 + i)z = (1 - i)\,\bar{z}$

$\Rightarrow \quad z=\frac{1-i}{1+i}\bar{z}=\frac{(1-i)(1-i)}{(1+i)(1-i)}\bar{z}=\frac{(1-i)^2}{(1-i^2)}\bar{z}$

$=\frac{1-2i+i^2}{1+1}\bar{z}=\frac{1-2i-1}{2}\bar{z}=-i\bar{z}$

$z = iz$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

26. Find the real value of 'a' for which $3i^3 - 2ai^2 + (1 - a)\,i + 5$ is real.

[Delhi Gov. QB 2022]

Ans. $3i^3 - 2ai^2 + (1 - a)i + 5$

$= -3i + 2a + (1 - a)i + 5$

$= (2a + 5) + i(1 - a - 3)$

$= (2a + 5) + i(-2 - a)$

Since, $3i^3 - 2ai^2 + (1 - a)i + 5$ is real.

$\therefore \text{Im}[3i^3 - 2ai^2 + (1 - a)i + 5] = 0$

$\Rightarrow \quad -2 - a = 0$

$\Rightarrow \quad a = -2$

Hence, the real value of a for which $3i^3 - 2ai^2 + (1 - a)i + 5$ is real is -2.

27. For what values a and b are $(1 - i)a + (1 + i)b$ and $1 - 3i$ equal?

Ans. Let $(1 - i)a + (1 + i)b = 1 - 3i$

$\Rightarrow \quad a - ia + b + ib = 1 - 3i$

$\Rightarrow \quad (a + b) + i(-a + b) = 1 - 3i$

So, the real and imaginary parts of the above complex numbers must be equal.

$\therefore$ We have,

$$a + b = 1 \quad \text{...(i)}$$
$$-a + b = -3 \quad \text{...(ii)}$$

On solving eq. (i) and (ii), we get

$$a = 2 \text{ and } b = -1.$$

28. **Find the conjugate of the following:**

(A) $\dfrac{1}{2+4i}$ **(B)** $(2+5i)^2$

Ans. (A) Let $z = \dfrac{1}{2+4i}$

$$= \frac{(2-4i)}{(2+4i)(2-4i)} = \frac{2-4i}{4-16i^2}$$

$$= \frac{2-4i}{20} = \frac{2}{20} - \frac{4}{20}i.$$

Then, the conjugate of z is $\bar{z} = \dfrac{2}{20} + \dfrac{4}{20}i.$

(B) We have,

$$(2+5i)^2 = (2)^2 + 2\times2\times5i + (5i)^2$$
$$= 4 + 20i - 25$$
$$= -21 + 20i$$

$\therefore$ Complex conjugate of

$$(2+5i)^2 = \overline{(-21+20i)}$$
$$= -21 - 20i$$

29. **If** $z_1 = \sqrt{2}(\cos 30° + i \sin 60°)$ **,** $z_2 = \sqrt{3}(\cos 60° + i \sin 30°)$**, then find the value of** $\text{Re}(z_1 z_2)$**.**

Ans. Given,

$$z_1 = \sqrt{2}(\cos 30° + i \sin 60°) \text{ and}$$
$$z_2 = \sqrt{3}(\cos 60° + i \sin 30°)$$

$$\therefore \quad z_1 z_2 = \left[\sqrt{2}(\cos 30° + i \sin 60°)\right] \times \left[\sqrt{3}(\cos 60° + i \sin 30°)\right]$$

$$= \sqrt{6}\,[(\cos 30° \cos 60° - \sin 30° \sin 60°) + i(\cos 30° \sin 30° + \sin 60° \cos° 60°)]$$

$$= \sqrt{6}\left[\cos(60°+30°) + i\left(\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right)\right]$$

$[\because \cos(A+B) = \cos A \cos B - \sin A \sin B]$

$$= \sqrt{6}\left[\cos 90° + i\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \sqrt{6}\left[0 + i\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= 0 + i\left(\frac{\sqrt{6}(\sqrt{3})}{2}\right) = \frac{0 + 3\sqrt{2}i}{2}$$

$\therefore$ $\text{Re}(z_1 z_2) = 0$

30. **Solve the equation** $|z| = z + 1 + 2i$**.**

[NCERT Exemplar]

Ans. We have, $|z| = z + 1 + 2i$

Putting $z = x + iy$, we get

$$|x + iy| = x + iy + 1 + 2i$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = (x+1) + i(y+2)$$

$$[\because |z| = \sqrt{x^2+y^2}]$$

On comparing real and imaginary parts, we get

$$\sqrt{x^2+y^2} = x + 1;$$

And $\quad 0 = y + 2$

$\Rightarrow \quad y = -2$

Putting this value of y in $\sqrt{x^2+y^2} = x + 1$ we get

$$\sqrt{x^2+(-2)^2} = x + 1$$

On squaring both sides, we get

$$x^2 + (-2)^2 = (x+1)^2$$
$$\Rightarrow \quad x^2 + 4 = x^2 + 2x + 1$$
$$\Rightarrow \quad 2x = 3$$
$$x = \frac{3}{2}$$

$$z = x + iy = \frac{3}{2} - 2i$$

31. **Find the modulus of** $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$**.**

Ans.

$$\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$$

$$= \frac{(1+i^2+2i) - (1+i^2-2i)}{1-i^2}$$

$$= \frac{4i}{2} = 0 + 2i.$$

Hence, $\left|\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}\right| = \sqrt{(0)^2 + (2)^2} = 2$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

32. $a+ib=\frac{(x+i)^2}{2x^2+1}$, prove that

$a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$. [Delhi Gov. SQP 2022]

Ans. Given, $a+ib=\frac{(x+i)^2}{2x^2+1}$

$$=\frac{x^2+i^2+2xi}{2x^2+1}$$

$$=\frac{x^2-1+i2x}{2x^2+1}$$

$$=\frac{x^2-1}{2x^2+1}+i\left(\frac{2x}{2x^2+1}\right)$$

On comparing real and imaginary parts, we obain

$$a=\frac{x^2-1}{2x^2+1} \text{ and } b=\frac{2x^2}{2x^2+1}$$

$$\therefore \quad a^2+b^2=\left(\frac{x^2-1}{2x^2+1}\right)^2+\left(\frac{2x}{2x^2+1}\right)^2$$

$$=\frac{x^4+1-2x^2+4x^2}{(2x^2+1)^2}$$

$$=\frac{x^2+1+2x^2}{(2x^2+1)^2}$$

$$=\frac{(x^2+1)^2}{(2x^2+1)^2}$$

$$\therefore \quad a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$$

Hence, proved.

33. Find the real numbers x and y such that $(x-iy)(3+5i)$ is the conjugate of $-6+24i$.

Ans. Given that $(x-iy)(3+5i)$ is the conjugate of $-6+24i$.

$\therefore \quad (x-iy)(3+5i)=-6+24i$

$\Rightarrow \quad 3x+5ix-3iy-5i^2y=-6+24i$

$\Rightarrow \quad (3x+5y)+i(5x-3y)=-6+24i$

So, the real and imaginary parts of the above complex numbers must be equal.

$3x+5y=-6$ ...(i)

$5x-3y=24$ ...(ii)

Performing eq. (i) × 3 + (ii) × 5, we get

$$9x+15y+25x-15y=-18+120$$

$$34x=102$$

$$x=\frac{102}{34}=3$$

Putting the value of x in eq. (i), we get

$$3\times3+5y=-6$$

$$5y=-6-9$$

$$5y=-15$$

$$y=\frac{-15}{5}=-3$$

Therefore the value of x and y are 3 and -3 respectively.

34. Find real value of x and y,

If $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$

Ans. We have,

$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$$

$$\frac{\{(1+i)x-2i\}(3-i)+\{(2-3i)y+i\}(3+i)}{(3+i)(3-i)}=i$$

$$\frac{\{x+i(x-2)\}(3-i)+\{2y+i(1-3y)(3+i)\}}{(3+i)(3-i)}=i$$

$$\frac{(3+3i-i-i^2)x-(6i-2i^2)+(6-9i+2i-3i^2)y+(3i+i^2)\}}{3^2-i^2}=i$$

$$(4+2i)x-(6i+2)+(9-7i)y+(3i-i)=i(9+1)$$

$$4x+2xi-6i-2+9y-7yi+3i-1=10i$$

$$\frac{(4x+9y-3)+i(2x-7y-3)}{10}=0+i$$

$$\frac{4x+9y-3}{10}=0 \text{ and } \frac{2x-7y-3}{10}=1$$

$4x+9y-3=0$ ...(i)

$2x-7y-13=0$ ...(ii)

On solving eq. (i) and (ii), we get

$x=3$ and $y=-1$

35. If the real part of $\frac{z+2}{z-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle. [NCERT Exemplar]

Ans. Let $z=x+iy$

Now,

$$\frac{z+2}{z-1}=\frac{x-iy+2}{x-iy-1}=\frac{[(x+2)-iy][(x-1)+iy]}{[(x-1)-iy][(x-1)+iy]}$$

$$= \frac{(x-1)(x+2)+y^2+i[(x+2)y-(x-1)y]}{(x-1)^2+y^2}$$

Given that the real part is 4.

$$\Rightarrow \frac{(x-1)(x+2)+y^2}{(x-1)^2+y^2}=4$$

$$\Rightarrow x^2+x-2+y^2 = 4(x^2-2x+1+y^2)$$

$\Rightarrow 3x^2+3y^2-9x+6=0$,

which represents a circle

Hence, locus of z is a circle.

36. **Find the number of non-zero integral solutions of the equation $|3^{1/2}-i|^x = 4^x$.**

Ans. $|3^{1/2}-i|^x = 4^x$

$$\Rightarrow [(\sqrt{(3^{1/2})^2+(-1)^2})]^x = 4^x$$

$$\Rightarrow (\sqrt{4})^x = 4^x$$

$$\Rightarrow 4^{x/2} = 4^x$$

$$\Rightarrow \frac{x}{2}=x$$

$$\Rightarrow x-\frac{x}{2}=0$$

$$\Rightarrow x=0$$

Hence, the given equation does not have a non-zero integral solution.

LONG ANSWER Type Questions (LA)

[**4** & **5** marks]

37. **If $x+iy=\sqrt{\frac{p+iq}{a+ib}}$, prove that**

$$(x^2+y^2)^2=\frac{p^2+q^2}{a^2+b^2}.$$

Ans. Given, $x+iy=\sqrt{\frac{p+iq}{a+ib}}$

$$\Rightarrow (x+iy)^2=\left(\frac{p+iq}{a+ib}\right)^2$$

[On squaring both sides]

$$\Rightarrow |(x+iy)^2|=\left|\frac{p+iq}{a+ib}\right|^2$$

[On taking modulus on both sides]

$$\Rightarrow |(x+iy)(x+iy)|=\left|\frac{p+iq}{a+ib}\right|^2$$

$$\Rightarrow |x+iy||x+iy|=\left|\frac{p+iq}{a+ib}\right|^2$$

$$\Rightarrow \sqrt{x^2+(y)^2}\sqrt{x^2+(y)^2}=\frac{\sqrt{p^2+(q)^2}}{\sqrt{a^2+(b)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2}\sqrt{x^2+y^2}=\frac{\sqrt{p^2+q^2}}{\sqrt{a^2+b^2}}$$

$$\Rightarrow x^2+y^2=\frac{\sqrt{p^2+q^2}}{\sqrt{a^2+b^2}}$$

$$\Rightarrow (x^2+y^2)^2=\frac{p^2+q^2}{a^2+b^2}$$

[On squaring both sides]

Hence, proved.

38. **Find real value of x and y, if**

$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i.$$

[Delhi Gov. QB 2022]

Ans. Consider the given expressions.

$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+1}{3-i}=i$$

$$\frac{x+(x-2)i}{3+i}+\frac{2y+(1-3y)i}{3-i}=i$$

$$\frac{(x+(x-2)i)(3-i)+(2y+(1-3y)i)(3+i)}{9-i^2}=i$$

$x(3-i)+i(x-2)(3-i)+2y(3+i)$
$+i(1-3y)(3+i)=(9+1)i$

$3x-ix+i(3x-ix-6+2i)+6y+2iy$
$+i(3+i-9y-3yi)=10i$

$3x-ix+3xi-i^2x-6i+2i^2+6y+2iy+3i$
$+i^2-9yi-3yi^2=10i$

$3x-ix+3xi+x-6i-2+6y+2iy+3i-1$
$-9yi+3y=10i$

$4x+9y-3+2xi-7yi-13i=0$

$4x+9y-3+(2x-7y-13)i=0$

On comparing real part and imaginary part, we get

$4x+9y-3=0$ ...(i)

$2x-7y-13=0$ ...(ii)

On solving both equations, we get

$x=3$

$y=-1$

Hence, the value of x, y is 3, −1.

39. If $\bar{x} - iy = \dfrac{(a+7)^2}{2a+i}$ find the value of $x^2 + y^2$.

Ans. Given, $x - iy = \dfrac{(a+7)^2}{2a+i}$ ...(i)

We know that if two complex numbers are equal, then their conjugates are also.

Taking conjugate of both sides, we get

$$x + iy = \frac{(a+7)^2}{2a-i} \quad \text{...(ii)}$$

Multiplying corresponding sides of both equations (i) and (ii), we get

$$(x - iy)(x + iy) = \frac{(a+7)^2}{2a+i} \times \frac{(a+7)^2}{2a-i}$$

$$\Rightarrow \quad x^2 - i^2y^2 = \frac{(a+7)^4}{(2a)^2 - i^2}$$

$$\Rightarrow \quad x^2 + y^2 = \frac{(a+7)^4}{4a^2+1}$$

40. Show that $\left|\dfrac{z-2}{z-3}\right| = 2$ represents a circle. Find its center and radius. [NCERT Exemplar]

Ans. We have $\left|\dfrac{z-2}{z-3}\right| = 2$

Putting $z = x + iy$, we get

$$\left|\frac{x+iy-2}{x+iy-3}\right| = 2$$

$$\Rightarrow \quad |x - 2 + iy| = 2\,|x - 3 + iy|$$

$$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

$$\Rightarrow \quad x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2)$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$\Rightarrow \quad \left(x - \frac{10}{3}\right)^2 + y^2 + \frac{32}{3} - \frac{100}{9} = 0$$

$$\Rightarrow \quad \left(x - \frac{10}{3}\right)^2 + (y-0)^2 = \frac{4}{9}$$

Hence, center of the circle is $\left(\dfrac{10}{3}, 0\right)$ and radius is $\dfrac{2}{3}$.